ON THE MOTION OF A PLANE PISTON IN A MEDIUM OF FINITE CONDUCTIVITY UNDER THE INFLUENCE OF AN ELECTROMAGNETIC FIELD

(O DVIZHENII PLOSKOGO PORSHNIA V SREDE S KONECHNOI PROVODIMOST'IU S UCHETOM VLIIANIIA ELEKTROMAGNITNOGO POLIA)

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We consider the problem of an infinitely conducting plane piston moving with constant velocity U in a medium of finite conductivity, which varies in the direction of motion of the piston (along the x-axis) according to the law

$$\sigma = sx^{-1} \tag{1}$$

where s is a constant in the region where the gas parameters are continuous. The problem for a medium with a constant finite conductivity was treated by Cole [1] with the assumption of small U. For the case of a shock wave with a jump in conductivity, the problem of a plane piston, taking into account the radiated electromagnetic waves, was solved in [2].

At the initial instant of time a plane piston starts to move with velocity U from the origin into a gas at rest with pressure p_0 and density ρ_0. At the initial instant let there be a constant magnetic field of intensity H_0, directed along the z-axis. A shock wave is propagated into the gas, abruptly changing the gas parameters. In front of the shock wave the initial electromagnetic field and the parameters of the medium will be disturbed by the passage of the electromagnetic waves. We wish to find the dependence of the gas velocity v, pressure p, density ρ, magnetic field H and the other characteristics on the coordinate x and the time t.

The system of magnetohydrodynamic equations, including the displacement current, have the form (c is the velocity of light)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \qquad \rho\frac{dv}{dt} = -\frac{\partial p}{\partial x} + \frac{1}{c}jH \qquad \frac{dp}{dt} = -\gamma p\frac{\partial v}{\partial x} + \frac{\gamma - 1}{\sigma}j^2 \tag{2}$$

$$\frac{\partial H}{\partial t} = -c\frac{\partial E}{\partial x} \qquad -\frac{\partial H}{\partial x} = 4\pi j\frac{1}{c} + \frac{1}{c}\frac{\partial E}{\partial t}, \qquad j = \sigma\left(E + \frac{v}{c}H\right)$$
$$(\mathbf{H}\cdot\mathbf{E} = 0)$$

The boundary conditions at the shock wave take the form

$$\{\rho(v-D)\} = 0, \qquad \{\rho v(v-D) + p\} = 0 \quad \left\{(v-D)\left(\frac{1}{\gamma-1}p + \frac{1}{2}\rho v^2\right) + pv\right\} = 0 \tag{3}$$
$$\left\{vH - \nu_m\frac{\partial H}{\partial x}\right\} = 0, \qquad \{H\} = 0, \qquad \nu_m = \frac{c^2}{4\pi\sigma}$$

Here D is the velocity of the shock wave, and the curly brackets denote the difference in quantities across the surface of discontinuity. At the piston we have the condition $v = U$.

From considerations of the theory of dimensional analysis [3] it follows that the sought-for characteristics depend only on one dimensionless variable, for which we take the quantity $\lambda = x(D_0 t)^{-1}$. We introduce the dimensionless functions

$$H = H_0 G(\lambda), \qquad E = H_0 F(\lambda), \qquad v = D_0 V(\lambda)$$
$$p = \rho_0 D_0^2 P(\lambda), \qquad \rho = \rho_0 R(\lambda), \qquad D = D_0\Delta \qquad (\Delta = \text{const})$$

Here D_0 is the velocity of the shock wave for $H_0 = 0$. In dimensionless form the system (2) reduces to a system of ordinary differential equations

$$-\lambda R' + (RV)' = 0, \qquad R(-\lambda^2 V' + \lambda VV') = -\lambda P' + qi \tag{4}$$
$$\lambda(V-\lambda)P' + \gamma\lambda PV' = \frac{(\gamma-1)}{\omega\delta}qi^2, \qquad \lambda G' = \delta^{-1}F', \qquad -\lambda G' = 4\pi i - \delta\lambda^2 F'$$
$$i = \omega(F - \delta VG), \ \omega = sc^{-1}, \qquad \delta = D_0 c^{-1}, \ q = H_0^2(D_0^2\rho_0)^{-1}$$

Condition (3) for $x = Dt$, i.e. for $\lambda = \Delta$, may also be written in dimensionless form. The condition on the piston becomes

$$V = \frac{U}{D_0} = V_n \quad \text{for } \lambda = V_n$$

Moreover, for $\lambda = \delta^{-1}$, i.e. at the electromagnetic wave front, the condition of continuity of the electric and magnetic fields must be satisfied.

The problem will be solved in the region between the electromagnetic wave and the shock wave (region *1*) and in the region between the shock wave and the piston (region *2*). The solutions found for regions *1* and *2* must be coupled by the conditions at the shock wave.

The problem is solved below by a perturbation method in a small parameter, which we take to be q. Such a method of solution clearly corresponds to either a weak initial magnetic field and a shock wave of moderate intensity, or the case of a moderate magnetic field, but a strong shock wave.

We seek a solution of the form

$$\begin{aligned} V &= V_{k0} + qV_{k1} + \ldots, & P &= P_{k0} + qP_{k1} + \ldots \\ R &= R_{k0} + qR_{k1} + \ldots, & G &= G_{k0} + qG_{k1} + \ldots \\ F &= F_{k0} + qF_{k1} + \ldots, & \Delta &= 1 + q\Delta_1 + \ldots \end{aligned} \tag{5}$$

where $k = 1$ in the region *1* and $k = 2$ in the region *2*. At the shock wave, the zero order functions must satisfy the conditions

$$\begin{gathered} R_{20}(V_{20} - 1) = -R_{10}, \qquad V_{20}R_{20}(V_{20} - 1) + P_{20} = P_{10} \\ (V_{20} - 1)\left(\frac{1}{\gamma - 1} P_{20} + \frac{1}{2} R_{20}V_{20}^2\right) + P_{20}V_{20} = -\frac{1}{\gamma - 1} P_{10} \\ 4\pi\delta V_{20} = \frac{1}{\omega_2}\left(\frac{dG}{d\lambda}\right)_2 - \frac{1}{\omega_1}\left(\frac{dG}{d\lambda}\right)_1 \quad G_{20} = G_{10} \end{gathered} \tag{6}$$

At the piston, i.e. for $\lambda = V_n$, we have the condition $V_{20} = V_n$.

We consider the solution of the problem in region *1*. Noting that $V_{10} = 0$, and substituting (5) into the system (4), we obtain for the zero order approximation

$$\lambda G_{10}' = \delta^{-1}F_{10}', \qquad -\lambda G_{10}' = 4\pi\omega_1 F_{10} - \delta\lambda^2 F_{10}' \tag{7}$$

The remaining equations of the system (4) are identically satisfied.

The solution of the system (7) has the form

$$F_{10} = c_{10}\left[\frac{1 - \delta\lambda}{1 + \delta\lambda}\right]^{2\pi\omega_1}, \quad G_{10} = -c_{10}4\pi\omega_1\int \lambda^{-1}(1 - \delta\lambda)^{2\pi\omega_1 - 1}(1 + \delta\lambda)^{-(2\pi\omega_1 + 1)}d\lambda \tag{8}$$

In particular, for $2\pi\omega_1 = 1$ we obtain

$$G_{10} = 2c_{10}\left[\ln\frac{(1 + \delta\lambda)}{\lambda} + \frac{\delta\lambda}{1 + \delta\lambda}\right] + c_{20}, \quad F_{10} = c_{10}\frac{1 - \delta\lambda}{1 + \delta\lambda} \tag{9}$$

We consider the solution of the problem in region 2. Substituting (5) into the system (4), we obtain for the zero order approximation the equations

$$-\lambda R_{20}' + (R_{20}V_{20})' = 0, \qquad R_{20}(V_{20} - \lambda)V_{20}' + P_{20}' = 0, \qquad (V_{20} - \lambda)P_{20}' + \gamma P_{20}V_{20}' = 0 \tag{10}$$

$$\lambda G_{20}' = \frac{1}{\delta} F_{20}', \qquad -\lambda G_{20}' = 4\pi i_{20} - \delta\lambda^2 F_{20}' \tag{11}$$

The system of Equations (10) is satisfied by the hydrodynamic solution

$$V_{20} = V_n, \qquad P_{20} = \text{const}, \qquad R_{20} = \text{const}$$

Equations (11) must be solved for V_{20} = const. From the system (11) we find

$$G_{20}' = c_{30}(1+\delta\lambda)^a(1-\delta\lambda)^b\lambda^{-1+4\pi\omega_2\delta V_{20}} \tag{12}$$
$$a = -(1+2\pi\omega_2+2\pi\omega_2\delta V_{20}), \qquad b = -1+2\pi\omega_2-2\pi\omega_2\delta V_{20}$$

In this way G_{20} is found by quadrature, and then F_{20} is determined from the second equation of the system (11). In the general case the function $G_{20}(\lambda)$ may not be written in a simple form. (For particular values of the parameters, for example, $4\pi\omega = 3$, $3\delta V_{20} = 1$, the integration can be carried out easily.)

However, by noting that $\lambda \leqslant 1$ in region 2, while $\delta = D_0/c$ is small for ordinary shock waves, we may obtain an approximate solution by neglecting the term $\delta\lambda$ in comparison with unity in Equations (12) and in the subsequent calculations. The conditions of continuity of the electromagnetic field across the shock wave and at the electromagnetic wave front are used for determination of the arbitrary constants appearing in the solution.

We consider the first approximation, i.e. we ascertain in what manner the presence of the initial magnetic field affects the motions of the gas to a first approximation.

In the regions *1* and 2, for R_{k1}, V_{k1}, P_{k1} we have the equations

$$\begin{aligned}
&R_{k0}V_{k1}' + (V_{k0}-\lambda)R_{k1}' = 0 \\
&R_{k0}(V_{k0}-\lambda)V_{k1}' + P_{k1}' = \Phi_{k1}(\lambda) \qquad \Phi_{k1}(\lambda) = \frac{\omega_k}{\lambda}(F_{k0}-\delta V_{k0}G_{k0})G_{k0} \\
&(V_{k0}-\lambda)P_{k1}' + \gamma P_{k0}V_{k1}' = \Phi_{k2}(\lambda) \qquad \Phi_{k2}(\lambda) = \frac{\gamma-1}{\delta\lambda}\omega_k(F_{k0}-\delta V_{k0}G_{k0})^2
\end{aligned} \tag{13}$$

Here Φ_{k1} and Φ_{k2} are known functions dependent on the zero approximation; $V_{10} = 0$, P_{10} and R_{10} are the constant dimensionless magnitudes of the pressure and density in the undisturbed medium. The conditions at the shock wave for the functions V_{k1}, R_{k1}, P_{k1} and G_{k1} are

$$\begin{gathered}
\{R_{k0}V_{k1} - R_{k0}\Delta_1 + R_{k1}V_{k0} - R_{k1}\} = 0 \\
\{(V_{k0}-1)(R_{k0}V_{k1} + V_{k0}R_{k1}) + V_{k0}R_{k0}(V_{k1}-\Delta_1) + P_{k1}\} = 0 \\
\Big\{(V_{k0}-1)\Big(\frac{1}{\gamma-1}P_{k1} + \frac{1}{2}R_{k1}V_{k0}^2 + R_{k0}V_{k0}V_{k1}\Big) + \\
+ (V_{k1}-\Delta_1)\Big(\frac{1}{\gamma-1}P_{k0} + \frac{1}{2}P_{k0}V_{k0}^2\Big) + P_{k0}V_{k1} + V_{k0}P_{k1}\Big\} = 0 \\
\Big\{4\pi\delta V_{k0}(G_{k0}'\Delta_1 + G_{k1}) + 4\pi\delta V_{k1}G_{k0} - \frac{1}{\omega_k}(G_{k0}''\Delta_1 + G_{k1}') + \frac{\Delta_1}{\omega_k}G_{k0}'\Big\} = 0 \\
\{G_{k1} + G_{k0}'\Delta_1\} = 0
\end{gathered} \tag{14}$$

On the piston for the second approximation we have the condition

$$V_{21} = 0 \quad \text{for } \lambda = U / D_0$$

The conditions on the electromagnetic wave for the first and the subsequent approximations are

$$P_{1m} = R_{1m} = G_{1m} = F_{1m} = 0 \qquad (m = 1, 2, 3 \ldots)$$

The solution of system (13) has the form

$$V_{11} = -\int_{1/\delta}^{\lambda} \frac{\Phi_{11} + \Phi_{12}}{\lambda^2 R_{10} - \gamma P_{10}} d\lambda, \qquad R_{11} = \int_{1/\delta}^{\lambda} \frac{R_{10}(\lambda\Phi_{11} + \Phi_{12})}{\lambda(\gamma P_{10} - \lambda^2 R_{10})} d\lambda$$

$$P_{11} = \int_{1/\delta}^{\lambda} \left[\Phi_{11} + \lambda R_{10} \frac{\lambda\Phi_{11} + \Phi_{12}}{\gamma P_{10} - \lambda^2 R_{10}}\right] d\lambda, \quad V_{21} = \int_{V_n}^{\lambda} \frac{(V_{20} - \lambda)\Phi_{21} - \Phi_{22}}{(V_{20} - \lambda)^2 R_{20} - \gamma P_{20}} d\lambda \qquad (15)$$

$$P_{21} = \int \left[\Phi_{21} - (V_{20} - \lambda) R_{20} \frac{(V_{20} - \lambda)\Phi_{11} - \Phi_{12}}{(V_{20} - \lambda)^2 R_{20} - \gamma P_{20}}\right] d\lambda + c_{21}$$

$$R_{21} = -\int \frac{R_{20}[(V_{20} - \lambda)\Phi_{11} - \Phi_{12}]\, d\lambda}{(V_{20} - \lambda)[(V_{20} - \lambda)^2 R_{20} - \gamma P_{20}]} + c_{31} \qquad (16)$$

The constants c_{21} and c_{31} are found together with Δ_1 from the system of three algebraic equations which are obtained after substitution of (15) and (16) into the first of three equations of the conditions (14). In this way the problem of determination of the functions in the first approximation is solved. The solution for subsequent approximations is obtained in an analogous fashion.

By this same method one may treat the problem of the motion of a plane piston with constant velocity in a detonating medium with the same law of variation of σ with x as was assumed above. The method used above may also be applied to the solution of the problem of propagation of a plane detonation wave including the effect of a magnetic field.

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